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Stress in Rotating Disks and Cylinders

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Abstract

The solution of the classic problem of stress in a rotating elastic disk or cylinder, as solved in standard texts on elasticity theory, has two features: dynamical equations are used that are valid only in an inertial frame of reference, and quadratic terms are dropped in displacement gradient in the definition of the strain. I show that, in an inertial frame of reference where the dynamical equations are valid, it is incorrect to drop the quadratic terms because they are as large as the linear terms that are kept. I provide an alternate formulation of the problem by transforming the dynamical equations to a corotating frame of reference of the disk/cylinder, where dropping the quadratic terms in displacement gradient is justified. The analysis shows that the classic textbook derivation of stress and strain must be interpreted as being carried out in the corotating frame of the medium.

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I. BACKGROUND

The U.S. Army is developing an electromagnetic gun (EMG) for battlefield applications. During the past few years, on a recurring basis, Dr. John Lyons (former ARL Director) and Dr. W. C. McCorkle (Director of U. S. Army Aviation and Missile Command) have requested that I look at some of the physics of the EMG. In a recent request, I was asked to look at stresses in the rotor of the EMG compulsator. The simplest physical model for a rotor is a rotating cylinder. Therefore, I spent some time looking at the physics of stresses in elastic rotating cylinders. The case of an elastic cylinder is a classic problem that is solved in many texts on linear elasticity [1–6]. However, when these derivations are examined closely, I found certain shortcomings in the treatments. In particular, when a body (cylinder) deforms during increasing angular velocity of rotation about its symmetry axis, the undeformed state (at zero angular velocity) and the deformed state (at finite angular velocity) are related by a large angle of rotation. When large angles of rotation coexist with deformations, it is well known that the quadratic terms (see Eq. (20)) in the definition of the strain tensor cannot be dropped, and this leads to complicated nonlinear differential equations. However, the problem of stresses in the rotating cylinder can be analyzed in a frame of reference (coordinate system) that is rotating with the cylinder. In this special non-inertial frame, the quadratic terms of the strain tensor can be dropped, and the stresses found by solving linear differential equations. The physics described above is not exposed in the standard treatments and this report is the subject of this explanation.

II. INTRODUCTION

The problem of stresses in rotating disks and cylinders is important in practical applications to rotating machinery, such as turbines and generators, and wherever large rotational speeds are used. The textbook problem of stresses in elastic rotating disks and cylinders, using the assumption of plane strain or plane stress, is published in classic texts, such as

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Love [1], Landau and Lifshitz [2], Nadai [3], Sechler [4], Timoshenko and Goodier [5], and Volterra and Gaines [6]. The standard approach presented in these texts has two characteristic features:

1. Newton's second law of motion is applied in an inertial frame of reference to derive dynamical equations for the continuum (see Eq. (1) below), and

2. quadratic terms in displacement gradient are dropped in the definition of the strain tensor (see Eq.(20) below).

In this paper, I show that, for a rotating elastic body, the second feature of the solution is inconsistent with the first: dropping the quadratic terms in the displacement gradient is an unjustified approximation in an inertial frame of reference. In what follows, I refer to the method that is employed in Ref. [1–6] as 'the standard method', and for brevity, I will refer to a cylinder as a generalization of both a disk and a cylinder.

The classic problem of stress in an elastic rotating cylinder is complex because the undeformed reference state of the body is the non-rotating state. The deformed state is one of steady-state rotation. The analysis of the problem must connect the non-rotating reference state to the rotating stressed/strained state. These two states are typically connected by large angles of rotation. When large angles of rotation are present, the quadratic terms in the displacement gradient cannot be dropped (in an inertial frame of reference) in the definition of the strain [7–9]. The problem of stress analysis when large-angle rotations are present is well known and has been discussed by a number of authors in general contexts, see for example [7–9]. However, large-angle rotations in the problem of a rotating elastic cylinder have not been dealt with in a technically correct manner, because quadratic strain gradient terms are incorrectly dropped in the 'standard method' [1–6].

In this work, I formulate the elastic problem of a rotating cylinder in a frame of reference that is corotating with the material. In this corotating frame, the quadratic terms in the displacement gradient can be dropped, and the resulting differential equations are linear and can be solved.

In section II, I review the 'standard method' of solution used in Ref. [1–6] and show that for a rotating cylinder the displacement gradient in an inertial frame of reference is of order unity, and therefore quadratic terms (in strain tensor definition) cannot be dropped when compared to the linear terms. Section III contains the bulk of the analysis. I describe the corotating systems of coordinates and the transformation of the velocity field to the corotating frame. I use the velocity transformation rules to transform the dynamical Eq. (1) from the inertial frame to the corotating frame (see Eq. (61) or (62)), where extra terms arise known as the centrifugal acceleration and the coriolis acceleration. In section IV, I write the explicit component equations for stress (in cylindrical coordinates) for the rotating elastic cylinder in its corotating frame. To display the resulting solution concretely, I derive the well-known formula for the stress in the rotating cylinder for the case of plane stress, as computed in the corotating frame. Stress is an objective tensor, i.e., stress is independent of observer motion [10,11], so the physical meaning of stress in the corotating frame is the same as in the inertial frame. Therefore, the stress field components in the corotating frame are equal to the stress field components in the inertial frame, see Eq. (36).

III. STANDARD SOLUTION METHOD

In the 'standard method' [1–6], the stress analysis of elastic rotating cylinders starts with the dynamical equations, which, in generalized curvilinear coordinates are given by [2,10,11]

$$\sigma^{kj}{}_{;j} + \rho f^k = \rho\, a^k \tag{1}$$

where σ^{kj} are the contravariant components of the stress tensor, f^k is the vector body force, and a^k is the acceleration vector. In Eq. (1), repeated indices are summed and the semicolon indicates covariant differentiation with respect to the coordinates. Expressed in terms of the velocity field in spatial coordinates, the acceleration is given by [10,11]

$$a^k = \frac{\partial v^k}{\partial t} + v^j v^k{}_{;j} \tag{2}$$

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where v^a is the velocity field, and the semicolon indicates covariant differentiation with respect to the coordinates, and $v^j v^k_{;j}$ is called the convective term. In Eq. (1), the stress σ^{kj}, acceleration a^k, and body force f^k, are generally time dependent. Equation (1) is derived by applying Newton's second law of motion to an element of the medium. Newton's second law is valid only in an inertial frame of reference, and consequently the validity of Eq. (1) is limited to inertial frames of reference.

In the 'standard method' of solution, Eq. (1) is applied by invoking an "effective body force", of magnitude equal to the centrifugal force in the rotating frame. In the inertial frame, there is actually no effective force (such as Coriolis or cetrifugal force). For the case of a body rotating about its principle axis, a more careful determination of the terms $f^k - a^k$ in Eq. (1) comes from setting the body force to zero (or setting equal to some applied force) and computing the material acceleration a^k for a given body motion. For a rigid body, or a uniform density elastic cylinder that is rotating about its axis of symmetry at a constant angular velocity ω_o, the Cartesian velocity field components are: $v^1 = -\omega_o y$; $v^2 = \omega_o x$, and $v^3 = 0$, where superscripts 1,2,3 indicate components on the Cartesian basis vectors associated with the x,y,z-axes (in the inertial frame). Corresponding to this velocity field, the cylindrical components of the acceleration field are given by

$$\bar{a}^k = \frac{\partial \bar{v}^k}{\partial t} + \bar{v}^b \bar{v}^k_{;b} = \left(-r\omega_o^2, 0, 0\right) \tag{3}$$

where I have chosen the z-axis as the symmetry axis and the bar over the components indicates that they are in the inertial frame of reference in cylindrical coordinates. For the case where there are no body forces, with the acceleration in Eq. (3), Eq. (1) in cylindrical coordinates leads to the three equations

$$\bar{\sigma}^{11}_{,1} + \bar{\sigma}^{12}_{,2} + \bar{\sigma}^{13}_{,3} + \frac{\bar{\sigma}^{11}}{r} - r\bar{\sigma}^{22} = -\rho r \omega_o^2 \tag{4}$$

$$\bar{\sigma}^{12}_{,1} + \bar{\sigma}^{22}_{,2} + \bar{\sigma}^{23}_{,3} + \frac{3}{r}\bar{\sigma}^{12} = 0 \tag{5}$$

$$\bar{\sigma}^{13}_{,1} + \bar{\sigma}^{23}_{,2} + \bar{\sigma}^{33}_{,3} + \frac{\bar{\sigma}^{13}}{r} = 0 \tag{6}$$

where the superscripts 1,2,3 enumerate tensor components on the r, ϕ, z coordinate basis vec-

tors respectively, in cylindrical coordinates and the commas indicate partial differentiation with respect to these coordinates.

For steady rotation at a uniform angular velocity ω_o, and assuming the absence of elastic waves, there is rotational symmetry about the z-axis so the stress components do not depend on azimuthal angle ϕ. Therefore, all derivatives with respect to ϕ are zero, leading to the equations:

$$\bar{\sigma}^{11}{}_{,1} + \bar{\sigma}^{13}{}_{,3} + \frac{\bar{\sigma}^{11}}{r} - r\bar{\sigma}^{22} = -\rho r\omega_o^2 \tag{7}$$

$$\bar{\sigma}^{12}{}_{,1} + \bar{\sigma}^{23}{}_{,3} + \frac{3}{r}\bar{\sigma}^{12} = 0 \tag{8}$$

$$\bar{\sigma}^{13}{}_{,1} + \bar{\sigma}^{33}{}_{,3} + \frac{\bar{\sigma}^{13}}{r} = 0 \tag{9}$$

I introduce physical components of stress, σ^{rr}, $\sigma^{\phi\phi}$, σ^{zz}, $\sigma^{r\phi}$, σ^{rz}, and $\sigma^{\phi z}$, with units of force per unit area and which are related to the tensor components $\bar{\sigma}^{11}$, $\bar{\sigma}^{22}$, $\bar{\sigma}^{33}$, $\bar{\sigma}^{12}$, $\bar{\sigma}^{13}$, and $\bar{\sigma}^{23}$, by [10,11]

$$\bar{\sigma}^{rr} = \bar{\sigma}^{11}, \tag{10}$$

$$\bar{\sigma}^{\phi\phi} = r^2\bar{\sigma}^{22} \tag{11}$$

$$\bar{\sigma}^{zz} = \bar{\sigma}^{33} \tag{12}$$

$$\bar{\sigma}^{r\phi} = r\bar{\sigma}^{12} \tag{13}$$

$$\bar{\sigma}^{rz} = \bar{\sigma}^{13} \tag{14}$$

$$\bar{\sigma}^{\phi z} = r\bar{\sigma}^{23} \tag{15}$$

Expressing Eq. (7)–(9) in terms of the physical components, I obtain the well-known equations valid in an inertial frame of reference [1–6],

$$\frac{\partial\bar{\sigma}^{rr}}{\partial r} + \frac{\partial\bar{\sigma}^{rz}}{\partial z} + \frac{\bar{\sigma}^{rr} - \bar{\sigma}^{\phi\phi}}{r} = -\rho r\omega_o^2 \tag{16}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\bar{\sigma}^{r\phi}\right) + \frac{1}{r}\frac{\partial\bar{\sigma}^{\phi z}}{\partial z} + \frac{3}{r^2}\bar{\sigma}^{r\phi} = 0 \tag{17}$$

$$\frac{\partial\bar{\sigma}^{rz}}{\partial r} + \frac{\partial\bar{\sigma}^{zz}}{\partial z} + \frac{\bar{\sigma}^{rz}}{r} = 0 \tag{18}$$

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Note that Eqs. (16)–(18) have been derived using Newtons's second law, and so they are valid only in an inertial frame of reference. In particular, Eqs. (16)–(18) are not valid in a rotating frame of reference.

When a rotating disk or cylinder is analyzed, the assumption of plane stress or plane strain is often made. In both cases, stresses must be related to strains by constitutive equations. For the simplest case of a homogeneous, isotropic, perfectly elastic body, the constitutive equations in curvilinear coordinates in an inertial frame can be written as [2,10,11]

$$\sigma^{ik} = \lambda\, e\, g^{ik} + 2\,\mu\, e^{ik} \tag{19}$$

where λ and μ are the Lamé material constants, e^{ik} are the contravariant strain tensor components, $e = e_a{}^a$ is the contraction of the strain tensor, and g^{ik} are the contravariant metric tensor components.

The Eulerian strain tensor e_{ik} is related to the displacement field u^i by [10,11]

$$e_{ik} = \frac{1}{2}\left(u_{i;k} + u_{k;i} + u_{m;i}u^m{}_{;k}\right) \tag{20}$$

In the 'standard method' of solution [1–6], the quadratic terms $u_{m;i}u^m{}_{;k}$ are dropped, which leads to linear equations that can be solved (for example, by using the Airy stress function [12]).

However, dropping the quadratic terms in Eq. (20) is not justified for a rotating body because these (dimensionless) terms $u_{m;i}$ are of order unity. To prove this assertion, it is sufficient to consider the limiting case of a rigid body in steady-state rotation at constant angular speed ω_o. The deformation mapping function gives the coordinates z^k (here taken to be Cartesian) of a particle at time t in terms of the particle's coordinates Z^k in some reference state (configuration) at time $t = t_o$:

$$z^k = z^k(Z^m, t) \tag{21}$$

so that $z^k(Z^m, t_o) = Z^k$. The deformation mapping function has an inverse, which I quote here for later reference

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$$Z^m = Z^m(z^k, t) \qquad (22)$$

Both z^k and Z^m refer to the same Cartesian coordinate system. The coordinates of a particle initially at Z^k at $t = t_o = 0$ rotating about the z-axis are given by the deformation mapping function

$$z^k = R^k{}_m Z^m \qquad (23)$$

where the orthogonal matrix $R^k{}_m$ is given by

$$R^k{}_m = \begin{pmatrix} \cos\omega_o t & \sin\omega_o t & 0 \\ -\sin\omega_o t & \cos\omega_o t & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (24)$$

The displacement vector field for this deformation mapping function is given by [13]

$$\mathbf{u} = u^m \mathbf{I}_m = (z^m - Z^m)\mathbf{I}_m = (\delta^m_k - \bar{R}^m{}_k)z^k \mathbf{I}_m \qquad (25)$$

where $\mathbf{I}_m$ are the unit Cartesian basis vectors and $\bar{R}^m{}_k$ is the transpose matrix that satisfies

$$\bar{R}^m{}_k R^k{}_l = \delta^m_l \qquad (26)$$

where $\delta^m_l = +1$ if $m = l$ and 0 if $m \neq l$. Therefore, from Eq. (25), it is clear that gradients of displacement $u_{m;k}$ appearing in Eq (20) are of order unity and therefore the quadratic terms $u_{m;i} u^m{}_{;k}$ cannot be dropped because they are not small. More specifically, the dropped terms in Eq. (20) vary in time between -2 and 0 (in Cartesian components):

$$\frac{1}{2} u_{m;i} u^m{}_{;k} = \begin{pmatrix} -1 + \cos\omega_o t & 0 & 0 \\ 0 & -1 + \cos\omega_o t & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (27)$$

The above calculation was done for a rigid body, but clearly, a similar error is introduced for elastic bodies. Therefore, in general, for a rotating elastic body, the quadratic terms in displacement gradients in Eq. (20) cannot be dropped [14].

In cylindrical components, the relation between the physical components (see Ref. [10,11]) of strain, $\bar{e}_{rr}$ and $\bar{e}_{\phi\phi}$, and physical components of the displacement field, (u_r, u_ϕ, u_z), is given by

$$\bar{e}_{rr} = \frac{\partial u_r}{\partial r} - \frac{1}{2}\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_\phi}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial r}\right)^2\right] \tag{28}$$

$$\bar{e}_{\phi\phi} = \frac{u_r}{r} - \frac{1}{2r^2}\left[(u_r)^2 + (u_\phi)^2\right] \tag{29}$$

(I use a bar over $\bar{e}_{rr}$ and $\bar{e}_{\phi\phi}$ to indicate that these are cylindrical components, and indices rr and $\phi\phi$ (as distict from 11 and 22) to indicate that these are physical components and not tensor components. See the Appendix and Table I and II for notation conventions.) In Eq. (28) and (29), I have assumed that there is no dependence on ϕ and z, so I have set derivatives with respect to these variables to zero.

In the 'standard method' of solving for the stress in a rotating cylinder [1–6], the quadratic terms in Eq. (28) and (29) are incorrectly dropped.

The straight forward approach to correctly studying the stresses in a rotating disk or cylinder, involves keeping the quadratic terms in displacement gradient in Eq. (20). However, this approach does not appear promising because it leads to insoluble nonlinear differential equations. In the next section, I approach the problem by using a transformation to a corotating frame of reference, in which dropping the quadratic terms can be justified for moderate angular velocity of rotation ω_o.

IV. TRANSFORMATION TO THE ROTATING FRAME

As discussed in the introduction, the problem of an elastic rotating cylinder is complicated because the unstressed/unstrained reference state is the non-rotating state, while the stressed (strained) state is rotating, and these two states are typically related by a large (time-dependent) angle. The analysis of the rotating disk or cylinder must relate the stresses in the rotating state to the reference configuration, which I take to be the non-rotating state. I define a transformation from an inertial frame of reference to the corotating frame of refer-

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ence of the cylinder. This transformation provides a relation between the rotating stressed state and the non-rotating reference configuration.

A. Coordinate Systems

Starting from an inertial frame of reference, S, defined by the Cartesian coordinates $z^k = (x, y, z)$, I make a transformation to a rotating frame of reference S'. The rotating frame will be corotating with the cylinder so that in this frame S' the azimuthal velocity field will be zero at all times. The transformation from an inertial system of coordinates to a rotating system of coordinates is most simply done using Cartesian coordinates. On the other hand, the assumed cylindrical symmetry of the problem begs for use of cylindrical coordinates. Hence I will make use of four systems of coordinates. In the inertial frame S, I have two systems of coordinates: a Cartesian system of coordinates $z^k = z_k = (x, y, z)$, and a cylindrical system of coordinates $x^i = (r, \phi, z)$. In the corotating frame of reference, S', I have a Cartesian system of coordinates $z'^k = (x', y', z')$ and a cylindrical coordinate system $x'^i = (r', \phi', z')$. These coordinates are summarized in Table I and the Appendix. I also introduce notation for tensor components in each of the four coordinate systems, see Table II . The Cartesian components of the stress tensor in the inertial frame S will be denoted by σ^{ik}. In the same inertial frame S, the cylindrical components of stress will be $\bar{\sigma}^{ik}$. The Cartesian components of stress in the corotating frame S' will have a prime, σ'^{ik}. In this same corotating frame, S', the cylindrical components of stress will be denoted by using a tilde, $\tilde{\sigma}^{ik}$.

From the vantage point of an inertial frame of reference, S, with Cartesian coordinates z^k, consider a cylinder whose symmetry axis is aligned and colocated with the coordinate z-axis. At time $t = -\infty$, take the cylinder to be non-rotating. Now assume that in the distant past, around the time $t \sim -T$, the cylinder begins a slow angular acceleration lasting a long time, on the order of $1/\epsilon$, where $1/\epsilon << T$. An example of such an angular acceleration function is

$$\omega(t) = \frac{1}{2}\omega_o\left[1 + \tanh(\epsilon(t+T))\right] \tag{30}$$

where I assume that $\tau \ll 1/\epsilon \ll T$ and τ is the longest time constant in the problem. This inequality states that the acceleration occurs slowly, $\tau \ll 1/\epsilon$, slower than any time scale in the problem, and that this acceleration occurs in the distant past, $1/\epsilon \ll T$, so that at $t = 0$, I have a steady-state situation of a cylinder rotating at constant angular speed ω_o. By slowly accelerating the cylinder, I avoid introducing modes of vibration. As the cylinder's angular velocity increases from $t = -\infty$, each particle comprising the cylinder moves along a spiral trajectory (with increasing radius). From the point of view of the inertial frame S, the stresses on a given element of the medium (particle) are such that they cause the particle to experience an acceleration, moving along the spiral path. At $t = 0$, the cylinder has achieved its maximum angular velocity ω_o. Due to the assumption of a perfectly elastic medium, at $t = 0$ the velocity field has zero radial component; all particles of the cylinder are moving azimuthally (in a plane perpendicular to the z-axis with zero radial component). The velocity field is that of a rigid body and the acceleration field is given by Eq. (3).

Now I introduce the corotating frame of reference, S', with the Cartesian coordinates z'_k, whose angular velocity of rotation is equal to that of the cylinder at all times. The coordinates $z_k \ (\equiv z^k)$ and $z'_k \ (\equiv z'^k)$ are related by

$$z'_i = A_{ik}(t)\, z_k \tag{31}$$

where the time dependent matrix $A_{ik}(t)$ is given by

$$A_{ik}(t) = \begin{pmatrix} \cos(\theta - \theta_o) & \sin(\theta - \theta_o) & 0 \\ -\sin(\theta - \theta_o) & \cos(\theta - \theta_o) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{32}$$

where θ is a function of time given by the integral of $\omega(t)$:

$$\theta(t) = \frac{\omega_o}{2\epsilon}\left[\epsilon(t+T) + \log\cosh(\epsilon t + \epsilon T) + \log 2\right] \tag{33}$$

and $\theta(0) = \theta_o$.

By construction, in the corotating frame S' the particles comprising the material are not rotating about the z'-axis; there is zero azimuthal component of the velocity field at all times. As the angular velocity $\omega(t)$ increases from $t = -\infty$, each particle comprising the cylinder experiences an increasing effective centrifugal force that displaces the particle to a larger radius. In this rotating frame S', there will (in general) also be a Coriolis force. However, in S', for moderate angular speed ω_o, the strain e_{ik} will be small, and the gradients of the displacement field will also be small. Consequently, dropping the quadratic terms $u_{m;i}u^m{}_{;k}$ in Eq. (20) will provide a good approximation to e_{ik}.

Note that the transformation that relates cylindrical components in inertial frame S and rotating frame S' is given by the identity matrix

$$\frac{\partial x^i}{\partial x'^k} = \frac{\partial x^i}{\partial z^a} \frac{\partial z^a}{\partial z'^m} \frac{\partial z'^m}{\partial x'^k} = \delta^i_k \tag{34}$$

Furthermore, the time dependent transformation between inertial cylindrical coordinates $x^i = (r, \phi, z)$ in S and corotating cylindrical coordinates $x'^i = (r', \phi', z')$ in S', is given by

$$r' = r$$
$$\phi' = \phi - (\theta(t) - \theta_o) \tag{35}$$
$$z' = z$$

where $\theta(t)$ is given by Eq. (33). The relation between cylindrical stress components $\bar{\sigma}^{ik}$ in the inertial frame S and cylindrical stress components $\tilde{\sigma}^{ik}$ in the corotating frame S' is

$$\bar{\sigma}^{ik}(x^n) = \frac{\partial x^i}{\partial x'^a} \frac{\partial x^k}{\partial x'^b} \tilde{\sigma}^{ab}(x'^n) = \tilde{\sigma}^{ik}(x'^n) \tag{36}$$

Of course the transformation in Eq. (36) must be used so that the components are referring to the same physical point in space having coordinates $x^n = (r, \phi, z)$ and $x'^n = (r', \phi', z')$, where the time-dependent relation between x^n and x'^n is given by Eq. (35).

The Eulerian strain tensor e_{ij} transforms in a more complicated manner. A general deformation is given by

$$x^i = x^i(X^k, t) \tag{37}$$

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where a particle at time $t = t_o$ in the reference configuration has (curvilinear) coordinates X^k, and in the deformed state at time t the particle has coordinates x^i (in the same curvilinear coordinate system). The Eulerian strain tensor $e_{ij}(X, x)$ depends on two points: X in the reference configuration and x in the deformed state. Consequently, under a general coordinate transformation to a moving frame, $x^i \to x'^i = h^i(x^k, t)$, the Eulerian strain is a two-point tensor, which transforms as a second rank tensor under transformation of deformed coordinates

$$x^i \to x'^i = h^i(x^k, t) \tag{38}$$

and transforms as a scalar under transformation of reference state coordinates

$$X^i \to X'^i = h^i(X^k, t_o) \tag{39}$$

so that [15–17]

$$\bar{e}_{mn}(X, x) = \tilde{e}_{ik}(X', x') \frac{\partial x^i}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} = \tilde{e}_{mn}(X', x') \tag{40}$$

where x and x', and X and X', are related by Eq. (38) and (39), and I used Eq. (34). Therefore, the cylindrical components of strain in the inertial frame S, $\bar{e}_{mn}(X, x)$, are equal to the cylindrical components of strain in the rotating frame S', $\tilde{e}_{ik}(X', x')$. Finally, since I assume rotational symmetry about the z-axis (and z'-axis) so that all physical quantities have no dependence on ϕ or Φ, which are the azimuthal coordinates of the point in the deformed state, $x = x^i = (r, \phi, z)$, and coordinates of the point in the reference configuration, $X = X^i = (R, \Phi, Z)$. Because of the nature of the transformation to the rotating frame in Eq. (35), Eq. (40) can be used with $r' = r$, $R' = R$, $z' = z$, and $Z' = Z$. Therefore, the tensor components in the corotating frame are identical to the components in the inertial frame.

B. Transformation of Lagrangean and Eulerian Velocities

The motion of a particle in the cylinder is given by the deformation mapping function. In the inertial frame S, using Cartesian coordinates, the motion of the particle is given by

Eq. (21), where the particle in the reference configuration at time $t = t_o$ has coordinates Z^m. The coordinates Z^m label the particle in the Lagrangean desription. Using the transformation to the rotating frame in Eq. (31), the motion of the particle with label Z^m with respect to the rotating S' frame Cartesian coordinates is given by

$$z'^k(Z^m, t) = A_{kj}(t)\, z^j(Z^m, t) \tag{41}$$

In discussing the transformation to the rotating system of coordinates, I must distinguish between the velocity of a given particle in the medium (the Lagrangean picture) and the velocity field (the Eulerian picture). The Lagrangean velocity $v_i(S; Z^m; t)$ of a given particle (whose coordinates in the reference configuration are Z^m) with respect to the inertial frame S is defined as the partial time derivative of that particle's z^i-coordinates, when holding Z^m constant

$$v_i(S; Z^m; t) = \frac{\partial z^i(Z^m, t)}{\partial t} \tag{42}$$

The Eulerian velocity field, $v_i(z^k, t)$, with respect to the frame S is a function of coordinates z^k and time t and is related to the Lagrangean (particle) velocity by

$$v_i(S; Z^m; t) = v_i(S; Z^m(z^k; t), t) = v_i(z^k, t) \tag{43}$$

where I used Eq. (22) to express the particle coordinate Z^m in terms of its position z^k at time t.

I can describe the same particle's velocity (whose coordinates in the (inertial frame) reference configuration are Z^m) with respect to the rotating frame of reference S'. The velocity of this particle with respect to the rotating frame S' is

$$v'_i(S'; Z^m; t) = \frac{\partial z'_i(Z^m, t)}{\partial t} = \frac{\partial}{\partial t}\left[A_{ik}(t)\, z_k(Z^m, t)\right] = \dot{A}_{ik}(t)\, z_k(Z^m, t) + A_{ik}(t)\, v_k(S; Z^m; t) \tag{44}$$

where I used Eq. (31) to express the particle's coordinates in terms of coordinates in the S frame and the dot on $\dot{A}_{ik}$ indicates differentiation with respect to time.

The components $v_i(S; Z^m; t)$ and $v'_i(S'; Z^m; t)$ represent physically distinct vectors (geometric objects). Each of these vectors can be expressed on the other basis. In particular, according to the standard transformation rules for vector components, I have

$$v'_i(S'; Z^m; t) = A_{im} \, v_m(S'; Z^m; t) \tag{45}$$

$$v'_m(S; Z^m; t) = A_{mi} \, v_i(S; Z^m; t) \tag{46}$$

Equations (45) and (46) are the standard tensor transformation rules for vector components under the coordinate transformation given in Eq. (31). In summary, I must distinguish between four (Cartesian component) velocities [18]:

$v_i(S; Z^m; t) = \dot{z}_i(Z^m, t) = $ components on z-axes of particle velocity with respect to S

$v_i(S'; Z^m; t) = $ components on z-axes of particle velocity with respect to S'

$v'_i(S'; Z^m; t) = \dot{z}'_i(Z^m, t) = $ components on z'-axes of particle velocity with respect to S'

$v'_m(S; Z^m; t) = $ components on z'-axes of particle velocity with respect to S

These four velocities are related. Multiplying Eq. (44) by $A_{im}(t)$, summing over index i and using the orthogonal matrix properties

$$A_{in}(t) \, A_{im}(t) = \delta_{nm} \tag{47}$$

$$A_{ni}(t) \, A_{mi}(t) = \delta_{nm} \tag{48}$$

leads to [18]

$$v_n(S'; Z^m; t) = v_n(S; Z^m; t) + \omega_{nk}(S', S) \, z_k(Z^m, t) \tag{49}$$

where $\omega_{nk}(S', S)$ is the Cartesian angular velocity tensor of frame S' with respect to frame S:

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$$\omega_{nk}(S', S) = A_{in}(t)\,\dot{A}_{ik}(t) \tag{50}$$

The tensor $\omega_{nk}(S', S)$ describes the time-dependent rotation of frame S' with respect to frame S. Similarly, using Eq. (21), substituting the inverse relation

$$z_i(Z^m, t) = A_{ni}(t)\,z'_n(Z^m, t) \tag{51}$$

carrying out the time differentiation, multiplying by A_{mi}, summing over i and use of the orthogonality relations in Eq. (48) leads to

$$v'_n(S; Z^m; t) = v'_n(S'; Z^m; t) + \omega'_{kn}(S', S)\,z'_k(Z^m, t) \tag{52}$$

where the angular velocity tensor components are expressed with respect to the S' frame Cartesian basis:

$$\omega'_{nm}(S', S) = A_{mi}\,\dot{A}_{ni} = A_{ni}\,A_{mk}\,\omega_{ik}(S', S) \tag{53}$$

Equation (49) and (52) are the well-known rules for transformating *particle* velocity to a rotating frame of reference [18].

Next, I derive the equation that relates the Cartesian components of the velocity field in S, $v_i(z^k, t)$, to the velocity field in S', $v'_i(z'^k, t)$. Equation (43) relates the velocity field in the S frame to the Lagrangean (particle) velocity. Similarly, the velocity field with respect to the S' frame is given by

$$v'_i(z'^k, t) = v'_i(S'; Z^m; t) = \frac{\partial\, z'^i(Z^m, t)}{\partial t} = v'_i(S'; Z^m(z^k, t); t) = v'_i(S'; Z^m(A_{nk}\, z'^n, t); t) \tag{54}$$

where I used the inverse relation $z^k = A_{nk}\,z'^n$. Using Eq. (43) and (54) in the left and right most terms in Eq. (44), I obtain a relation between the velocity fields in frames S and S'

$$v'_i(z'^k, t) = \dot{A}_{ik}(t)\,z_k + A_{ik}(t)\,v_k(z^n, t) \tag{55}$$

In Eq. (55), the coordinates z'^k and z^n are related by Eq. (31). Multiplying Eq. (55) by A_{im}, summing over index, i, using the inverse transformation in Eq. (51) and

$$\omega_{jk}(S', S) = A_{mj} A_{nk} \omega'_{mn}(S', S) \tag{56}$$

leads to

$$v_j(z_m, t) = A_{ij}(t) v'_i(z'_n, t) - A_{kj}(t) \omega'_{kn}(S', S) z'_n \tag{57}$$

Eq. (57) is the desired rule for transformation of the Eulerian velocity field from the inertial frame S to the rotating frame S'. Note that the right side of Eq. (57) depends only on S' frame coordinates z'_n and the left side depends on S frame coordinates z_m. Furthermore, the velocity field components on the left side of Eq. (57) are taken on the (Cartesian) inertial frame S basis vectors, and on the right side all components are expressed on the (Cartesian) rotating frame S' basis vectors.

C. Dynamical Equation in the Rotating Frame

In what follows, I transform the momentum balance Eq. (1) to the corotating system of coordinates S'. For simplicity, I do this transformation using Cartesian coordinates for both the inertial frame S and corotating frame S'. I use the transformation of the velocity field given in Eq. (57) to compute the terms that appear in Eq. (1). Taking the gradient of the velocity in Eq. (57)

$$\frac{\partial v_j(z_m, t)}{\partial z_k} = A_{nk} A_{ij} \frac{\partial v'_i(z'_n, t)}{z'_n} - A_{mk} A_{nj} \omega'_{nm}(S', S) \tag{58}$$

where I used the chain rule for differentiation $\frac{\partial}{\partial z_k} = A_{mk} \frac{\partial}{\partial z'_m}$ since z_k and z'_m are related by Eq. (41). Next, I compute the time derivative of the velocity that occurs in Eq. (1) and I express the right side in terms of S' frame components and coordinates:

$$\frac{\partial v_i(z_m, t)}{\partial t} = \dot{A}_{mi} v'_m + A_{li} \omega'_{nm} z'_m \frac{\partial v'_l}{\partial z'_n} + A_{mi} \frac{\partial v'_m}{\partial t} - \dot{A}_{ri} \omega'_{rn} z'_n - A_{mi} \dot{\omega}'_{mn} z'_n - A_{li} \omega'_{nm} \omega'_{ln} z'_m \tag{59}$$

where I have omitted the frame labels of the angular velocity and the coordinate arguments in the velocity. The divergence of the stress transforms as a vector under the transformation to the rotating frame

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$$\frac{\partial \sigma_{ik}}{\partial z_k} = A_{ai} \frac{\partial \sigma'_{ab}}{\partial z'_b} \tag{60}$$

Substituting Eqs. (57)–(60) into the inertial-frame momentum balance Eq. (1) and simplifying, leads to the dynamical equation for the (Cartesian) stress tensor in the rotating frame S'

$$\frac{1}{\rho} \frac{\partial \sigma'_{ik}}{\partial z'_k} = \frac{\partial v'_i}{\partial t} + v'_n \frac{\partial v'_i}{\partial z'_n} + \left(\omega'_{im} \omega'_{mn} - \frac{\partial \omega'_{in}}{\partial t} \right) z'_n - 2\omega'_{im} v'_m \tag{61}$$

where σ'_{ik} are the stress components in Cartesian coordinates z'_k, v'_n is the Eulerian velocity field that depends on z'_k and t, and the (Cartesian) components of the angular velocity tensor in S' are given by Eq. (53). In Eq. (61), all repeated subscripts are summed. Note that all velocities that appear in Eq. (61) refer to the corotating frame S' and that all tensor components are taken on the S' frame Cartesian basis vectors. The first two terms in Eq. (61) are the acceleration (including the convective term) as seen in the corotating system of coordinates. The third term $\omega'_{im} \omega'_{mn} z'_n$ is the centrifugal acceleration. The fourth term, $\dot{\omega}'_{in} z'_n$ is the angular acceleration. The last term, $-2\omega'_{in} v'_n$ is the Coriolis acceleration.

The dynamical Eq. (61) is a tensor equation; the quantities σ'_{kn}, ω'_{mn}, and v'_n, are Cartesian tensors. Under orthogonal transformations from one Cartesian system to another, Eq. (61) is covariant: it has the same form. The group of symmetry operations may be extended to transformations between curvilinear coordinates by writing Eq. (61) in a manifestly covariant form as:

$$\frac{1}{\rho} \, \tilde{\sigma}^{ik}{}_{;k} = \frac{\partial \tilde{v}^i}{\partial t} + \tilde{v}^n \, \tilde{v}^i{}_{;n} + 2\tilde{\omega}_n{}^i \, \tilde{v}^n + \left(\tilde{\omega}_m{}^i \tilde{\omega}_n{}^m + \frac{\partial \tilde{\omega}_n{}^i}{\partial t} \right) \tilde{\zeta}^n \tag{62}$$

where the tilde over each tensor indicates that the components are taken on generalized curvilinear coordinate (such as cylindrical) basis vectors in the rotating frame of reference S'. Equation (62) is general; it is valid for all motions and all materials. Equation (62) expresses Newton's law for a continuous medium in an arbitrary rotating frame of reference that is defined by a general (Cartesian) angular velocity tensor $\omega_{ik}(S', S) = A_{li}(t) \dot{A}_{lk}(t)$, which relates the inertial frame S and rotating frame S', with coordinates related by Eq. (31).

The quantities $\tilde{\zeta}^n$ are the contravariant components of the position vector in curvilinear coordinates x'^m, which are related to the Cartesian position vector components z'^k by

$$\tilde{\zeta}^m = \frac{\partial x'^m}{\partial z'^k} \, z'^k \tag{63}$$

The rotating frame curvilinear components of stress, velocity and angular velocity, $\tilde{\sigma}^{ab}$, $\tilde{v}^a$, and $\tilde{\omega}_a{}^b$, are related to their rotating frame Cartesian components, σ'_{ij}, v'^k, and ω'_{ij}, by:

$$\tilde{\sigma}^{ab} = \frac{\partial x'^a}{\partial z'^i} \frac{\partial x'^b}{\partial z'^j} \, \sigma'_{ij} \tag{64}$$

$$\tilde{v}^a = \frac{\partial x'^a}{\partial z'^k} \, v'^k \tag{65}$$

$$\tilde{\omega}_a{}^b = \frac{\partial z'^i}{\partial x'^a} \frac{\partial x'^b}{\partial z'^j} \, \omega'_{ij} \tag{66}$$

where z'^k and x'^a are the Cartesian and curvilinear coordinates in the rotating frame S'. Note that the partial derivative with respect to coordinates z'^k in Eq. (61) has been replaced by a covariant derivative with respect to the curvilinear coordinates x'^k in Eq. (62). See Tables I and II for summary of the notation. Identification of the meaning of the various terms in Eq. (62), such as the Coriolis acceleration and centrifugal accelleration is clear from comparison with Eq. (61).

V. ROTATING CYLINDER EQUATIONS IN COROTATING COORDINATES

In order to obtain the explicit equations for a rotating cylinder from Eq. (62), I need to compute $\tilde{v}^n$, $\tilde{\omega}_m{}^n$, and ζ^n. As described previously, I take the reference configuration of the cylinder to be the stationary cylinder at $t = -\infty$ in the inertial frame S. I assume that the cylinder experiences a slow angular acceleration (such as given by Eq. (30)) that lasts approximately time $\Delta t = -1/\epsilon$, and has peak magnitude at $t = -T$, with $1/\epsilon << T$. Equation (62) is general; it is valid for all motions and all materials. In what follows, I restrict my remarks to a perfectly elastic cylinder.

In the inertial frame S, as the cylinder increases its angular velocity, the material particles of the cylinder move outward in a spiral path, with some motion in the z-direction. In the

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inertial frame S, the contravariant components of the velocity field in cylindrical coordinates are

$$\bar{v}^i = (\bar{v}^1, \omega(t), \bar{v}^3) \tag{67}$$

where $\bar{v}^1$ and $\bar{v}^3$ are the radial and z-components of velocity and where the azimuthal component $\omega(t)$ is given by Eq. (30). I obtain the velocity components in cylindrical coordinates in the corotating frame S' as follows: first, transform $\bar{v}^i$ to inertial frame Cartesian components v^i using the standard vector transformation rule between Cartesian and cylindrical coordinates. Next, use Eq. (57) to transform the inertial frame Cartesian components v^i to the rotating S' frame Cartesian velocity field v'^i. Finally, use the standard tensor transformation rules, between Cartesian and cylindrical coordinates (both in the rotating frame), to transform the velocity field from Cartesian components v'^i to cylindrical (rotating frame) components $\bar{v}'^i$, where I made use of the angular velocity components

$$\omega_{ik}(S', S) = A_{li}\,\dot{A}_{lk} = \omega(t) \begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \omega'_{ik}(S', S) = A_{im}\,A_{kn}\,\omega_{mn}(S', S) \tag{68}$$

Following this procedure, I obtain the velocity components in cylindrical coordinates in the corotating frame S'

$$\tilde{v}^i = \left(\bar{v}^1,\, 0,\, \bar{v}^3\right) \tag{69}$$

where the azimuthal component of velocity is zero for all time, by construction of the corotating frame S', as expected.

In the corotating frame S', there is particle motion around the time $t \approx -T$. However, at $t = 0$ the particles have reached their new (deformed) steady-state positions and motion has ceased; the velocity field is given by

$$\tilde{v}^k = (0, 0, 0) \tag{70}$$

which is the velocity field of a rigid body. Since the velocity field is zero at $t = 0$, the Coriolis acceleration term in Eq. (62) does not contribute in steady-state rotation.

Using Eq. (66), the angular velocity tensor in cylindrical coordinates in the corotating frame S' is

$$\tilde{\omega}_a{}^b = \begin{pmatrix} 0 & \omega(t)/r' & 0 \\ -r'\omega(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{71}$$

At $t = 0$, the time derivative of the angular velocity tensor is zero. Using this fact, and Eq. (70) and (71), the stress Eqs. (62) in the corotating frame are given by

$$\tilde{\sigma}^{11}{}_{,1} + \tilde{\sigma}^{13}{}_{,3} + \frac{\tilde{\sigma}^{11}}{r'} - r'\tilde{\sigma}^{22} = -\rho r' \omega_o^2 \tag{72}$$

$$\tilde{\sigma}^{12}{}_{,1} + \tilde{\sigma}^{23}{}_{,3} + \frac{3}{r'}\tilde{\sigma}^{12} = 0 \tag{73}$$

$$\tilde{\sigma}^{13}{}_{,1} + \tilde{\sigma}^{33}{}_{,3} + \frac{\tilde{\sigma}^{13}}{r'} = 0 \tag{74}$$

Equations (72)–(74) are the equations satisfied by the stress tensor in cylindrical components at $t = 0$, in steady-state rotation in the corotating frame S'. In terms of physical components (see Eq. (10)–(15)), Eq. (72)–(74) become:

$$\frac{\partial \tilde{\sigma}^{rr}}{\partial r'} + \frac{\partial \tilde{\sigma}^{rz}}{\partial z'} + \frac{\tilde{\sigma}^{rr} - \tilde{\sigma}^{\phi\phi}}{r'} = -\rho r' \omega_o^2 \tag{75}$$

$$\frac{\partial}{\partial r'}\left(\frac{1}{r'}\tilde{\sigma}^{r\phi}\right) + \frac{1}{r'}\frac{\partial \tilde{\sigma}^{\phi z}}{\partial z'} + \frac{3}{r'^2}\tilde{\sigma}^{r\phi} = 0 \tag{76}$$

$$\frac{\partial \tilde{\sigma}^{rz}}{\partial r'} + \frac{\partial \tilde{\sigma}^{zz}}{\partial z'} + \frac{\tilde{\sigma}^{rz}}{r'} = 0 \tag{77}$$

Note that Eq. (75)–(77) in the corotating frame S' have the same form as Eq. (16)–(18) in the inertial frame S. However, the key point is that the corotating frame Eq. (75)–(77) have a distinct advantage: the quadratic strain gradient terms in the definition of the strain in Eq. (20) can be dropped because they are small in the corotating frame S'. The same is not true in the inertial frame S.

To proceed with the solution in the corotating frame, the constitutive Eq. (19) (in inertial frame S) must be transformed to the corotating frame S' by taking cylindrical components in the S frame and using Eq. (34) to obtain an expression of the same form as in Eq. (19) but in the corotating frame S'. In this transformation, the Lamé constants are treated as

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invariants. So the constitutive relations in the corotating frame S' are the same as in the inertial frame S.

In the corotating frame S', dropping the quadratic terms in displacement gradients, which relate the displacement field to strain, is justified since in the rotating frame S' these terms can be considered small for moderate angular velocity. Therefore, the solution in the corotating frame S' can proceed in an analogous way to that of the 'standard method', but I have not made the (incorrect) approximation of dropping quadratic displacement gradient terms in the inertial frame.

The stress tensor is objective, so the stress in the rotating frame has the same meaning as in the inertial frame, see Eq. (36). The boundary conditions on the stress tensor components in the corotating frame are the same as in the inertial frame, due to the objectivity of stress tensor. Alternatively, one can verify that the boundary conditions on the stress in the rotating frame are the same as in the inertial frame [20].

VI. PLANE STRESS SOLUTION

The solution of the problem of stress in a rotating cylinder in the corotating frame of reference now follows. The solution in the corotating frame parallels the solution in the 'standard method' [1-6], except that the incorrect approximation of dropping the quadratic strain gradient terms in the inertial frame is avoided.

I assume that in the inertial frame S, the cylinder is rotating at angular velocity ω_o and has radius b. Under the assumption of plane stress [5], where

$$\tilde{\sigma}^{zz} = \tilde{\sigma}^{\phi z} = \tilde{\sigma}^{rz} = 0 \tag{78}$$

with boundary condition of zero stress on the long peripheral surface:

$$\tilde{\sigma}^{rr}|_{r'=b} = 0 \tag{79}$$

As mentioned above, stress is an objective tensor [10,11], so the physical meaning of the boundary conditions in Eq. (78) and (79) in the corotating frame S' are the same as the

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physical meaning of the analogous conditions in the inertial frame (as used, for example by Timoshenko [5]). Also, as shown above, I may take the (transformed) linear elastic relations in the corotating frame to be of the same form as in the inertial frame:

$$\tilde{e}^{rr} = \frac{1}{E} \left(\tilde{\sigma}^{rr} - \nu \tilde{\sigma}^{\phi\phi} \right) \tag{80}$$

$$\tilde{e}^{\phi\phi} = \frac{1}{E} \left(\tilde{\sigma}^{\phi\phi} - \nu \tilde{\sigma}^{rr} \right) \tag{81}$$

where Young's modulus E and Poisson's ratio ν are related to the Lamè constants by

$$E = \frac{\mu \left(3\lambda + 2\mu \right)}{\lambda + \mu} \tag{82}$$

$$\nu = \frac{\lambda}{2 \left(\lambda + \mu \right)} \tag{83}$$

where the Lamè constants are treated as invariant scalars in the transformation. The elastic relations in Eq. (80) and (81) are in the corotating frame S'. They can be obtained from the linear elastic relations in the inertial frame S, Eq. (19), by using the transformations to the corotating frame in Eq. (36) and (40), and the coordinate transformation in Eq. (35). In doing the transformation of the elastic relations to the corotating frame, I am assuming that the tensor that enters in the inertial frame elastic relations in Eq. (19) is the Eulerian strain tensor given in Eq.(20) and the quadratic terms have not been dropped. (As discussed earlier, dropping the quadratic terms in the displacement gradients is done in the corotating frame.)

For small displacements in corotating frame S', the gradients of the displacement vector in frame S' can be assumed small—for moderate angular velocity ω_o, so the quadratic terms in the gradient of the displacement can be neglected. Therefore, in the corotating frame S', I take the relation between the radial component of the displacement vector, $\tilde{u}$, and the physical components of strain, $\tilde{e}_{rr}$ and $\tilde{e}_{\phi\phi}$, to be (compare with Eq. (28) and (29))

$$\tilde{e}_{rr} = \frac{\partial \tilde{u}}{\partial r'} \tag{84}$$

$$\tilde{e}_{\phi\phi} = \frac{\tilde{u}}{r'} \tag{85}$$

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where the tilde on $\tilde{u}$ indicates that the radial component of displacement field is taken in the corotating frame S' and the prime on r' indicates that the cylindrical radial coordinate is in the corotating frame S', see the transformation in Eq. (35).

Substituting Eq. (84) and (85) into Eq. (80) and (81), leads to relations between the physical components of stress and the radial displacement field

$$\tilde{\sigma}^{rr} = \frac{E}{1-\nu^2} \left(\frac{\partial \tilde{u}}{\partial r'} + \nu \frac{\tilde{u}}{r'} \right) \tag{86}$$

$$\tilde{\sigma}^{\phi\phi} = \frac{E}{1-\nu^2} \left(\frac{\tilde{u}}{r'} + \nu \frac{\partial \tilde{u}}{\partial r'} \right) \tag{87}$$

Substituting these relations into Eq. (75) leads to a differential equation for the displacement in the rotating frame S'

$$r'^2 \frac{\partial^2 \tilde{u}}{\partial r'^2} + r' \frac{\partial \tilde{u}}{\partial r'} - \tilde{u} = -\frac{1-\nu^2}{E} \rho \omega_o^2 r'^3 \tag{88}$$

The general solution is [5]

$$\tilde{u} = \frac{1}{E} \left[(1-\nu) C_1 r' - (1+\nu) C_2 \frac{1}{r'} - \frac{1-\nu^2}{8} \rho \omega_o^2 r'^3 \right] \tag{89}$$

Substitution of this solution into Eq. (86) and (87) I obtain

$$\tilde{\sigma}^{rr} = C_1 + C_2 \frac{1}{r'^2} - \frac{3+\nu}{8} \rho \omega_o^2 r'^2 \tag{90}$$

$$\tilde{\sigma}^{\phi\phi} = C_1 - C_2 \frac{1}{r'^2} - \frac{1+3\nu}{8} \rho \omega_o^2 r'^2 \tag{91}$$

The stresses at $r' = 0$ must remain finite, so I take $C_2 = 0$. Applying the boundary condition on the long peripheral surface, Eq. (79) leads to $C_1 = (3+\nu) \rho \omega_o^2 b^2 / 8$ and the stresses

$$\tilde{\sigma}^{rr} = \frac{3+\nu}{8} \rho \omega_o^2 (b^2 - r'^2) \tag{92}$$

$$\tilde{\sigma}^{\phi\phi} = \frac{1}{8} \rho \omega_o^2 \left[(3+\nu)b^2 - (1+3\nu)r'^2 \right] \tag{93}$$

The physical stress components in Eq. (92) and (93) are in the corotating frame S'. However, due to the transformation between the corotating frame and the inertial frame in Eq. (36), and the coordinate transformation in Eq. (35), the corotating frame components in Eq. (92) and (93) are equal to the inertial frame components of stress. Using the expressions in the rotating frame, such as Eq. (75)—(77), expressions for plane strain and other boundary conditions can be derived for rotating cylinders, disks and annular rings, see Ref. [1–6].

VII. SUMMARY

The classic problem of stress in rotating disks or cylinders is important in applications to turbines, generators, and whenever large rotational speeds exist. The textbook problem of stress in perfectly elastic disks or cylinders is solved in standard texts [1–6]. The 'standard method' of solution begins with Eq. (1) and drops terms that are quadratic in strain gradient in the definition of the strain, see Eq. (20). Equation (1) is valid only in an inertial frame of reference, since it is derived from Newton's second law of motion, which itself is only valid in an inertial reference frame.

In this work, I have shown that dropping the terms quadratic in the displacement gradient (in Eq. (20)) is incorrect in the inertial frame in which Eq. (1) is applied in the 'standard method' of solution [1–6]. I provide an alternative formulation of the rotating elastic cylinder problem in a frame of reference that is corotating with the cylinder. In this corotating frame, I derive the dynamical equation for the stress (see Eq. (61) or (62)) and I show that terms quadratic in the displacement gradient can be dropped because they are small (for moderate angular speed of rotation). This analysis in the corotating frame shows that the 'standard method' of solution [1–6] should be interpreted as being carried out in the corotating frame of reference of the cylinder.

Furthermore, when stresses are computed in rotating disks or cylinders composed of materials that have more complex constitutive equations, such as elastic-plastic or viscoelastic behavior, one must carefully justify dropping the quadratic terms in displacement gradients. If dropping these terms cannot be justified, then the problem can be analyzed in a rotating frame, using the derived Eq. (61) or (62). Another practical application of the stress Eq. (62) in the rotating frame is to study elastic waves in bodies during rotation, where coriolis effects may play a role.

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APPENDIX A: CONVENTIONS

I specify tensor components on coordinate (non-holonomic) basis vectors using numerical indices, 1,2,3, such as $\bar{\sigma}^{12}$. For physical components, which have the dimensions associated with that quantity, I use lettered indices, such as $\bar{\sigma}^{r\phi}$. In addition, I must distinguish between four coordinate systems: Cartesian and cylindrical coordinates in the inertial frame S and Cartesian and cylindrical in the corotating frame S'. I use $z^k = (x, y, z)$ and $x^k = (r, \phi, z)$ for Cartesian and cylindrical coordinates in inertial frame S, respectively. In corotating frame S', I use $z'^k = (x', y', z')$ and $x'^k = (r', \phi', z')$ for Cartesian and cylindrical coordinates, respectively. For distinguishing components in these four coordinate systems, I use an additional mark as follows: absence of mark and a bar, for Cartesian and cylindrical components in inertial frame S, respectively. For components in the corotating frame S', I use a prime and a tilde, for Cartesian and cylindrical components, respectively. See Table I and II.

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TABLES

TABLE I. Coordinates

	Inertial Frame S	Rotating Frame S'
Cartesian	$z^k = z_k$	$z'^k = z'_k$
Cylindrical	x^k	x'^k

TABLE II. Tensor Components

	Inertial Frame S	Rotating Frame S'
Cartesian	$\sigma^{ab}(z^k) = \sigma_{ab},\ e_{ik},\ \omega_a{}^b$	$\sigma'^{ab}(z'^k) = \sigma'_{ab},\ e'_{ik},\ \omega'_a{}^b$
Cylindrical	$\bar{\sigma}^{ab}(x^k),\ \bar{e}_{ik},\ \bar{\omega}_a{}^b$	$\tilde{\sigma}^{ab}(x'^k),\ \tilde{e}_{ik},\ \tilde{\omega}_a{}^b$

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13. ABSTRACT *(Maximum 200 words)*

The solution of the classic problem of stress in a rotating elastic disk or cylinder, as solved in standard texts on elasticity theory, has two features: dynamical equations are used that are valid only in an inertial frame of reference, and quadratic terms are dropped in displacement gradient in the definition of the strain. The author shows that, in an inertial frame of reference where the dynamical equations are valid, it is incorrect to drop the quadratic terms because they are as large as the linear terms that are kept. The author provides an alternate formulation of the problem by transforming the dynamical equations to a corotating frame of reference of the disk/cylinder, where dropping the quadratic terms in displacement gradient is justified. The analysis shows that the classic textbook derivation of stress and strain must be interpreted as being carried out in the corotating frame of the medium.

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